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NOTE ON THE RANK OF A SYMMETRICAL MATRIX.

By J. H. M. Wedderburn.

In the preceding paper, Professor Dickson has given a very simple proof of Kronecker's Theorem on the rank of a symmetrical matrix. The proof of this theorem which is given in this note, while not so elementary as Dickson's, shows very clearly the algebraic nature of the result.

Let $A = (a_{ij})$ be a symmetrical matrix of order n and rank r and

$$\varphi(x_1, \cdots, x_n) = \Sigma a_{ij} x_i x_j$$

the corresponding quadratic form. If the determinant of A is zero, there exists a point $(x_{11}, \cdots, x_{1n})$ such that

$$\sum_{j=1}^{n} a_{ij} x_{ij} = 0 \qquad (i = 1, 2, \cdots, n).$$

It is obvious geometrically, and not difficult to show algebraically, that there is an orthogonal matrix

$$B = (b_{ij}), \qquad |b_{ij}| \neq 0,$$

which transforms the point $(x_1, \cdots x_n)$ into $(1, 0, \cdots 0)$; and if the variables in the quadratic form are transformed by this orthogonal matrix, the matrix of the new form is $B'AB$ which is also symmetrical and has the same rank as A. From the way in which B was chosen, the coefficients in the first column of $B'AB$ are all zero and, as it is symmetrical, the same is true of the first row. The transformation therefore expresses φ in terms of $n - 1$ variables at most. This process may be repeated $n - r$ times the final result being a symmetrical matrix, $C'AC$, in which the first $n - r$ rows and columns are zero, while the determinant formed from the remaining elements is not zero. Since C, being the product of orthogonal matrices, is itself orthogonal, we have

$$C'AC = C^{-1}AC;$$

but the characteristic determinants $|A - \lambda|$ and $|C^{-1}AC - \lambda|$ are identical, and the coefficient of $(-1)^s\lambda^{n-s}$ in $|A - \lambda|$ is the sum of the principal minors of A of order s; therefore, since in $|C^{-1}AC - \lambda|$ this sum is zero if $s > r$ and not zero if $s = r$, we have the theorem:

In a symmetrical matrix of rank r the sum of the principal minors of order r is not zero.

It follows immediately that at least one principal minor of order r is not zero.

The same result is readily derived for skew symmetrical matrices and for matrices of the form $A + \lambda A'$.

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